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Original article

Fractional Calculus Study Comparison Between Riemann-Liouville Fractional and Caputo Fractional

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Abstract
This work aims to explore the two most common definitions of fractional calculus, the commonly used Riemann-Liouville and Caputo, elucidating examples, and spotlighting the comparison between Riemann-Liouville and Caputo fractional calculus by highlighting different examples.
Keywords. Fractional Calculus, Gamma Function, Riemann-Liouville's Fractional Derivative and Integration, Caputo's Fractional Derivative.

Introduction

The idea of integration and differentiation is familiar to all who have studied elementary calculus, $f(x) = x^3$. Thereafter, integrating $f(x)$ *to the* 1^{st} order results in $\int f(x)dx = \frac{1}{4}x^4 + c_1$, and integrating the same function into the 2^{nd}order results in $\int[\int f(x)dx]dx = \frac{1}{20}x^5 + c_1\,x + c_2$. Likewise, $\frac{d}{dx}f(x) = 3x^2$, and $\frac{d^2}{dx^2}f(x) = 6x$. But what if we wanted to integrate our function $f(x)$ to the $\frac{1}{2th}$ order, or discover its $\frac{1}{2th}$ order derivative?[5].

It's a famed belief that the idea of fractional calculus dates to the 17^{th}century when Leibniz reflected the tolerance of using non-integer order in calculus [3,6,8,12].

In 1695, in a letter to French mathematician Guillaume de L'Hôpital, Gottfried Leibniz wrote ''can the meaning of derivatives with integer order be propagated to derivatives with non-integer order?''.

L'Hôpital replied to Leibniz by formulating another question: "What will happen if $n = \frac{1}{2}$?".

In another letter, Leibniz replied ''It will lead to a paradox, from which one day useful consequences will be drawn [12].

In the same year, the derivative of general order was detailed in the letter from Leibniz to J.Bernoulli [1,7,10,12,13,14].

In 1819, Sylvestre F.Lacroix become the first mathematician to publish a paper that mentioned a fractional derivative [2], by suppose $y = x^m$, m integer, found the n^{th} derivative

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}x^{m-n} \;, m \geq n$$

by using Gamma function

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n} \,,$$

letting $m = 1$, and $n = \frac{1}{2}$, he obtained

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

In 1823, Niels Henrik Abel employed fractional calculus to solve an integral equation, showing that fractional operations, and their application to the solution of the Tautochrom problem. This paper, we explore, In section 2, we define the Gamma function, we also define the Riemann-Liouville's fractional derivative-integral and define Caputo fractional derivative, and provide numerous examples, In section 3, examines key comparison between Riemann -Liouville and Caputo fractional derivatives, by theorem relation between Riemann-Liouville fractional and Caputo derivative, to spotlight the different examples for showing the distinction.

Preliminaries

This section introduces the fundamental mathematical concepts and definitions required to establish a clear understanding of fractional calculus, starting with the Gamma function as a primary tool for constructing fractional operators.

Definition 2.1: Gamma function [12]

The gamma function, denoted by $\Gamma(z)$, is defined on a complex plane, i.e

$$\Gamma(z) = \int_0^\infty e^{-t}\,t^{z-1}dt\,, \qquad (Re\;z > 0) \tag{1}$$

with the relations: $\Gamma(z+1) = z\Gamma(z)$, and $\Gamma(z) = (z-1)!\,,\; z \in \mathbb{N}$

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The Gamma function is fundamental in fractional calculus as it generalizes the factorial function to non-integer orders, providing the necessary framework for defining fractional derivatives.

Definition 2.2: Riemann-Liouville's fractional derivative [11]

Suppose that $n \in \mathbb{N}$. The Riemann -Liouville fractional derivative ${}^{R}D_t^{\alpha}$ of order $\alpha \in (n-1, n]$ for a function v is defined as

$$ {}^{R}D_t^{\alpha} v(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\zeta)^{n-\alpha-1} \; v(\zeta) d\zeta \,, t > 0 \tag{2} $$

Definition 2.3: Riemann-Liouville's fractional integral [12]

Suppose α is a real non-negative number and $f(t)$ is piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval $[0, \infty)$. The Riemann-Liouville's fractional integral of $v(t)$ of order $\alpha > 0$, for $\zeta > 0$ is defined as

$$ {}_{a}I_t^{\alpha} v(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \frac{v(\zeta)}{(t-\zeta)^{1-\alpha}} d\zeta \,, \alpha > 0 \\ f(t) \qquad\qquad\qquad\quad , \;\; \alpha = 0 \end{cases} \tag{3} $$

Listed below are some of the basic properties of Riemann-Liouville's fractional derivative and integral [12]

i. For a function $v(t)$ and $\alpha, \beta \geq 0$, then
$ {}_{a}I_t^{\alpha} \, {}_{a}I_t^{\beta} \, v(t) = {}_{a}I_t^{\alpha+\beta} \, v(t)$
$ {}_{a}I_t^{\alpha} \, {}_{a}I_t^{\beta} \, v(t) = \; {}_{a}I_t^{\beta} \, {}_{a}I_t^{\alpha} \; v(t)$

ii. For all scalars λ and μ, we have:
$D_t^{\alpha}(\lambda v(t) + \mu g(t)\,) = \lambda D_t^{\alpha} v(t) + \mu D_t^{\alpha} g(t)$.

iii. ${}_{a}D_t^{0} v(t) = v(t)$

iv. ${}_{a}D_t^{\alpha} \, {}_{a}I_t^{\alpha} v(t) = v(t)$

v. ${}_{a}D_t^{\alpha}\big(v(t)g(t)\big) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \, {}_{a}D_t^{k} v(t) \, {}_{a}D_t^{\alpha-k} g(t)$

Example 2.4

Let the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(t) = 100$

Let $n = 1$. Then, the Riemann-Liouville derivative of fractional order $\alpha = \frac{1}{2}$ is

$$ {}_{RL}D^{\frac{1}{2}} f(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dt} \int_0^t (t-\zeta)^{\frac{-1}{2}} \;\; 100 \, d\zeta $$

$$ {}_{RL}D^{\frac{1}{2}} f(t) \; = \frac{100 \, t^{\frac{-1}{2}}}{\Gamma\left(\frac{1}{2}\right)} $$

Since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Then,

$$ {}_{RL}D^{\frac{1}{2}} f(t) = \frac{100}{\sqrt{t\pi}} $$

Example 2.5

Riemann-Liouville's integral of the function $f(t) = ln9$ of order $\alpha = \frac{5}{2}$

Using the equation (3), we get

$$ I^{\frac{5}{2}} \, ln9 = \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^t (t-\zeta)^{\frac{5}{2}-1} \;\; ln9 \, d\zeta $$

$$ I^{\frac{5}{2}} \, ln9 \;\; = \;\; \frac{ln9}{\Gamma\left(\frac{5}{2}\right)} \; \int_0^t (t-\zeta)^{\frac{3}{2}} \; d\zeta = \frac{ln9}{\Gamma\left(\frac{5}{2}\right)} \; \frac{2}{5} \;\; t^{\frac{5}{2}} $$

Since $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\pi$

Then,

$$ I^{\frac{5}{2}} l \; n9 = \frac{8t^{\frac{5}{2}}}{15\sqrt{\pi}} \;\; ln9 $$

Definition 2.6

The error function, denoted as $\operatorname{erf}(x)$, is defined by the integral

$$ \operatorname{erf}(x) = \tfrac{2}{\sqrt{\pi}} \textstyle\int_0^x e^{-t^2} dt \tag{4} $$

Example 2.7

Riemann-Liouville fractional integral of the function $f(x) = e^{2t}$ of order $\alpha = \frac{1}{2}$ is

$$ I_t^{\frac{1}{2}} e^{2t} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \frac{e^{2\zeta}}{\sqrt{t-\zeta}} d\zeta $$

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Let $u^2 = 2(t-\zeta)$

When $\zeta = 0$, $u = \sqrt{2t}$, when $\zeta = t, u = 0$

$$I_t^{\frac{1}{2}} e^{2t} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{2t}}^{0} \frac{e^{2\left(t-\frac{u^2}{2}\right)}}{\sqrt{u^2/2}} (-u)\, du$$

$$I_t^{\frac{1}{2}} e^{2t} = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\sqrt{2t}} e^{-u^2}\, du$$

by equation (4), we get.

$$I_t^{\frac{1}{2}} e^{2t} = \frac{1}{\sqrt{2}} e^{2t} erf\left(\sqrt{2t}\right)$$

Definition 2.8: Caputo's fractional derivative [9]

If $v: \mathbb{R}_+ \to \mathbb{R}$ and $\alpha \in (n-1, n), n \in \mathbb{N}$, then

$$ {}_0^c D^\alpha v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-1-\alpha} \ v^{(n)}(\zeta) d\zeta \tag{5}$$

Listed below are some of the basic properties of Caputo fractional derivative [12]

i. ${}_aD_t^\alpha I_t^\alpha\, v(t) = v(t)$
ii. ${}_aD_t^k v(t) = v^k(t)$
iii. ${}_aD_t^0 v(t) = v(t)$
iv. ${}_aI_t^\alpha\, {}_aD_t^\alpha v(t) = v(t) - \sum_{k=0}^{m-1} \frac{{}_aD_t^k v(\zeta)}{k!} (t-\zeta)^k$
v. For all scalars λ and μ, Caputo fractional derivative satisfies the linearity property as follows: $D_t^\alpha(\lambda v(t) + \mu g(t)) = \lambda D_t^\alpha v(t) + \mu D_t^\alpha g(t)$
vi. ${}_aD_t^\alpha\big(v(t)g(t)\big) = \sum_{k=0}^{\infty} \binom{\alpha}{k}\, {}_aD_t^k v(x)\, {}_aD_t^{\alpha-k} g(t)$

Example 2.9

Caputo fractional derivative of function $f(t) = 5$ directly from equation (5), we get

$$ {}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t 0 (t-\zeta)^{n-1-\alpha} d\zeta = 0$$

Theorem 2.10 [9] :The Caputo derivative is a linear operator, i.e. for any $a, b \in \mathbb{R}$

$$ {}^cD^\alpha(af(t) + bg(t)) = a\, {}^cD^\alpha f(t) + b\, {}^cD^\alpha g(t)$$

Proof. Suppose ${}^cD^\alpha f(t)$ and ${}^cD^\alpha g(t)$ are the Caputo derivatives of functions f and g. Then

$$ {}^cD^\alpha\big(af(t) + bg(t)\big) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\big(af(t) + bg(t)\big)^{(n)}}{(t-\zeta)^{\alpha-n+1}} d\zeta$$

$$= \frac{1}{\Gamma(n-\alpha)} (a \int_0^t \frac{f^{(n)}(t)}{(t-\zeta)^{\alpha-n+1}} d\zeta + b \int_0^t \frac{g^{(n)}(t)}{(t-\zeta)^{\alpha-n+1}} d\zeta)$$

$$= \frac{1}{\Gamma(n-\alpha)} a \int_0^t \frac{f^{(n)}(t)}{(t-\zeta)^{\alpha-n+1}} d\zeta + \frac{1}{\Gamma(n-\alpha)} b \int_0^t \frac{g^{(n)}(t)}{(t-\zeta)^{\alpha-n+1}} d\zeta$$

$$= a\, {}^cD^\alpha f(t) + b\, {}^cD^\alpha g(t)$$

Theorem 2.11: (Relation between Riemann-Liouville and Caputo fractional derivative) [4]

Let $t > 0, \alpha \in \mathbb{R}, n-1 \le \alpha < n \in \mathbb{N}$. Then the following relation between Riemann and Caputo satisfies

$$ {}^cD^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0)$$

Proof. Using the definition (2.2), the integrating by parts, we get

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\zeta)^{n-\alpha-1} f(\zeta) d\zeta$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[\frac{-(t-\zeta)^{n-\alpha} f(\zeta)}{n-\alpha} \Big|_0^t + \int_0^t \frac{f'(\zeta)(t-\zeta)^{n-\alpha}}{n-\alpha} d\zeta \right]$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[\frac{t^{n-\alpha} f(0)}{n-\alpha} + \int_0^t \frac{f'(\zeta)(t-\zeta)^{n-\alpha}}{n-\alpha} d\zeta \right]$$

$$= \frac{d^n}{dt^n} \left[\sum_{k=0}^{n-1} \frac{t^{n+k-\alpha}}{\Gamma(n+k-\alpha+1)} + \frac{1}{\Gamma(2n-\alpha)} \int_0^t (t-\zeta)^{2n-\alpha-1} f^{(n)}(\zeta)\, d\zeta \right]$$

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$$= \sum_{k=0}^{n-1} \frac{t^{k-\alpha} f^{(k)}(0)}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-\alpha-1} f^{(n)}(\zeta)\, d\zeta$$

$$D^\alpha(f(t)) = \sum_{k=0}^{n-1} \frac{t^{k-\alpha} f^{(k)}(0)}{\Gamma(k-\alpha+1)} + {}^cD^\alpha f(t)$$

i.e

$${}^cD^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0).$$

Comparison between Riemann-Liouville and Caputo's Fractional derivative

The main goal in this section is to make a comparison between Riemann-Liouville and Caputo Fractional as mentioned in the paper [12].

1. The derivative of the Riemann-Liouville constant is non zero, but Caputo's fractional derivative of a constant function is zero, as shown in Examples (2.4), (2.9).
2. The relationship between Riemann-Liouville and Caputo's Fractional derivative is given by the identity, as proven in Theorem 2.11

$${}_0^cD_t^\alpha f(t) = {}_0^{RL}D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0) \tag{6}$$

3. For Riemann-Liouville's initial value problem

$${}_0D_t^\alpha h(t) = F(h(t),t), m-1 \le \alpha < m \in \mathbb{Z}^+, t > 0 \tag{7}$$
$${}_0D_t^{\alpha-k} h(t)|_{t=0} = h_0^k, k = 1,2,\dots,m$$

While Caputo's sense initial value problem.

$${}_0D_t^\alpha h(t) = F(h(t),t), m-1 < \alpha \le m \in \mathbb{Z}^+, t > 0 \tag{8}$$
$$h^k(0) = h_0^k, k = 0,1,2,\dots,m$$

4. Laplace transform for Riemann-Liouville's Fractional

$$\mathcal{L}[{}_0D_t^\alpha h(t)] = S^\alpha \mathcal{L}[{}_0D_t^\alpha h(t)] - \sum_{k=0}^{m-1} S^k\, {}_0D_t^{\alpha-k-1} h(t)|_{t=0}, \tag{9}$$
$$m-1 \le \alpha < m \in \mathbb{Z}^+, t > 0$$

While Laplace Caputo

$$\mathcal{L}[{}_0D_t^\alpha h(t)] = S^\alpha \mathcal{L}[{}_0D_t^\alpha h(t)] - \sum_{k=0}^{m-1} S^{\alpha-k-1} h^k(0), \tag{10}$$
$$m-1 < \alpha \le m \in \mathbb{Z}^+, t > 0$$

5. For Riemann-Liouville's fractional derivative, Let $\alpha_1, \alpha_2 \in \mathbb{R}^+$, then

$${}_0D_t^{\alpha_1}\, {}_0D_t^{\alpha_2} g(t) \ne {}_0D_t^{\alpha_1+\alpha_2} g(t) \ne {}_0D_t^{\alpha_2}\, {}_0D_t^{\alpha_1} g(t)$$

Example 3.2

$${}_0D_t^{\frac{3}{2}}\, {}_0D_t^2 c = 0,\ {}_0D_t^2\, {}_0D_t^{\frac{3}{2}} c = \frac{c}{\Gamma\left(\frac{3}{2}\right)} t^{-\frac{3}{2}},\ {}_0D_t^{\frac{7}{2}} c = \frac{c}{\Gamma\left(\frac{3}{2}\right)} t^{-\frac{7}{2}}$$

And for Caputo's fractional derivative, Let $\alpha_1, \alpha_2 \in \mathbb{R}^+$, $\alpha_1 + \alpha_2 \le 1$. Then

$${}_0D_t^{\alpha_1}\, {}_0D_t^{\alpha_2} g(t) = {}_0D_t^{\alpha_1+\alpha_2} g(t) = {}_0D_t^{\alpha_2}\, {}_0D_t^{\alpha_1} g(t)$$

Example 3.3

$${}_0D_t^{0.6}\, {}_0D_t^{0.3}(t) = 1.0513 t^{0.1} = {}_0D_t^{0.3}\, {}_0D_t^{0.6}(t) = {}_0D_t^{0.9}(t)$$

In the case where $\alpha_1 + \alpha_2 > 1$, we obtain, ${}_0D_t^{\alpha_1}\, {}_0D_t^{\alpha_2} g(t) \ne {}_0D_t^{\alpha_1+\alpha_2} g(t)$

Example 3.4

$${}_0D_t^{0.7}\, {}_0D_t^{0.6} = \frac{1}{\Gamma(0.7)} t^{-0.3};\ {}_0D_t^{1.3}(t) = 0.$$

Then

$${}_0D_t^{0.7}\, {}_0D_t^{0.6} \ne {}_0D_t^{1.3}(t)$$

The following table summarizes the key differences between Riemann-Liouville and Caputo fractional derivatives, focusing on their mathematical properties and physical applications:

The summary provided in (Table 1) highlights the practical advantages of the Caputo operator, particularly in its ability to handle constant functions and physical initial conditions in a manner analogous to classical calculus.

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Table 1. Comparison between Riemann-Liouville and Caputo fractional derivatives

Comparison Criteria	Riemann-Liouville (RL)	Caputo Derivative
Derivative of a Constant	Non-zero (Yields $t^{-\alpha}$ terms)	Identically Zero (Matches classical calculus)
Initial Conditions	Fractional-order limits (Difficult to measure)	Integer-order derivatives (Physical meaning)
Origin Behavior ($t = 0$)	Potential singularity at the origin	Generally well-defined and regular
Fundamental Application	Theoretical mathematics and series	Engineering and physical modeling
Intuitive Consistency	Less aligned with traditional derivatives	High consistency with standard calculus

Conclusion

We can see from the above discussion that the Caputo and Riemann-Liouville derivatives do not coincide. Hence, this paper provided a comparison between Riemann-Liouville and Caputo; we presented some important results of the solution of basic functions using the fractional derivative and the fractional integral. Furthermore, we exhibited the relation between the Riemann-Liouville and Caputo fractional derivative theory.

Conflict of interest. Nil

References

1. Baleanu B, Diethelm K, Scalas E, Trujillo JJ. Fractional calculus: models and numerical methods. Vol. 2. Singapore: World Scientific Publishing; 2015.
2. Ross B, editor. Fractional calculus and its applications. Proceedings of the International Conference held at the University of New Haven; 1974 Jun; New Haven (CT). Berlin: Springer-Verlag; 1975.
3. Diethelm K. The analysis of fractional differential equations. Berlin: Springer; 2010.
4. Fakhouri W. Fractional differentiation [thesis]. Hebron: Palestine Polytechnic University; 2017.
5. Kimeu JM. Fractional calculus: definitions and applications [master's thesis]. Bowling Green (KY): Western Kentucky University; 2009.
6. Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. New York (NY): John Wiley and Sons, Inc.; 1993.
7. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam: Elsevier Science B.V.; 2006. (North-Holland Mathematics Studies; vol. 204).
8. Podlubny I. Fractional differential equations. San Diego (CA): Academic Press; 1999. (Mathematics in Science and Engineering; vol. 198).
9. Sikora B. Remarks on the Caputo fractional derivative. Minut. 2023;(5):76-84.
10. Diethelm K, Ford NJ. Analysis of fractional differential equations. J Math Anal Appl. 2002;265(2):229-48.
11. Guswanto BH, Andini LR, Triyani. On conformable, Riemann-Liouville, and Caputo fractional derivatives. Bull Appl Math Math Educ. 2022;2(2):59-64.
12. Pandey HR, Phaijoo GR, Gurung DB. A comprehensive study of fractional-order derivative and their interplay with basic functions. JNMS. 2024.
13. Ortigueira MD, Tenreiro Machado JA. What is a fractional derivative? J Comput Phys. 2015;293:4-13.
14. Velasco MP, Usero D, Jimenez S, Vazquez L, Poletti JLV, Mortazavi M. About some possible implementations of the fractional calculus. Mathematics. 2020;8:893.

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