Original article

A Simplified Adaptive Runge-Kutta Method with Competitive Performance for Ordinary Differential Equations

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Abstract
This paper presents a simplified and computationally efficient adaptive Runge–Kutta method for solving ordinary differential equations (ODEs). The proposed approach enhances the classical fourth-order Runge–Kutta (RK4) scheme by incorporating an intelligent step-doubling error estimation strategy, enabling reliable adaptive step-size control without relying on embedded Runge–Kutta pairs or complex Butcher tableaus. By comparing one full RK4 step with two half-steps, the method obtains a robust local error estimate that balances accuracy and computational cost while preserving implementation simplicity. The performance of the proposed adaptive RK4 method is rigorously evaluated against well-established solvers, namely RK45, DOP853, and the backward differentiation formula (BDF), as implemented in the SciPy library. Benchmark tests are conducted on three representative problems: the Van der Pol oscillator, a logistic growth model, and a nonlinear oscillator. Numerical results demonstrate that the proposed method consistently achieves high computational efficiency while maintaining accuracy within prescribed tolerances ranging from 10^{-4} to 10^{-6}. In particular, the method attains peak efficiencies exceeding 10^4 steps per second across all test cases. These results indicate that the proposed adaptive RK4 algorithm offers a practical and competitive alternative for general-purpose ODE solving, especially in applications where a balance between numerical accuracy, computational efficiency, and algorithmic simplicity is essential.
Keywords. Adaptive Runge–Kutta methods, Ordinary differential equations, Step-size control, Numerical integration, Error estimation.

Introduction

The numerical solution of ordinary differential equations (ODEs) constitutes a fundamental component of modern computational science, with extensive applications in physics, engineering, biological systems, and applied mathematics. For most nonlinear ODEs, closed-form analytical solutions are rarely available, rendering numerical integration methods indispensable. Among these methods, Runge–Kutta (RK) schemes—and in particular the classical fourth-order Runge–Kutta method (RK4)—remain widely used due to their favorable balance between numerical accuracy, stability, and implementation simplicity [1]. Despite its widespread adoption, the classical RK4 method employs a fixed step size, which can lead to inefficiencies when solving problems characterized by rapidly varying dynamics or multiple time scales. In such situations, an excessively small step size increases computational cost, whereas a large step size may compromise numerical accuracy. Adaptive step-size control techniques have therefore been developed to dynamically adjust the integration step based on local error estimates, thereby improving both efficiency and reliability [2,3]. Numerous studies have demonstrated that adaptive methods significantly outperform fixed-step approaches in a broad range of applications.

Most adaptive Runge–Kutta solvers rely on embedded methods, in which two schemes of different orders share function evaluations to estimate the local truncation error. Well-known examples include RK45 and DOP853, which are widely implemented in modern scientific computing libraries [4]. Although these solvers offer excellent accuracy and robustness, they often require complex Butcher tableaus and introduce additional implementation overhead. Furthermore, high-order and implicit formulations—such as diagonally implicit Runge–Kutta (DIRK) and embedded DIRK methods—are primarily designed for stiff systems and may be unnecessarily sophisticated for many non-stiff problems of moderate accuracy requirements [5, 6, 7].

Recent research has focused on improving the stability properties, efficiency, and applicability of adaptive Runge–Kutta methods. These efforts include the development of A-stable and stiffly accurate schemes [8], optimized solvers for higher-order ordinary differential equations [3,9], and parallel implementations aimed at reducing computational and energy costs [5]. While these advances have significantly expanded the capabilities of adaptive solvers, they have also increased algorithmic complexity, which can limit their practical usability in applications where simplicity and ease of implementation are essential. Consequently, there remains a practical gap between highly sophisticated adaptive Runge–Kutta solvers and classical fixed-step methods such as RK4. In particular, there is a lack of adaptive RK4-based algorithms that provide competitive computational efficiency while maintaining minimal algorithmic complexity and straightforward implementation. Addressing this gap is especially important for general-purpose ODE solving, where robustness, efficiency, and simplicity must be balanced.

Motivated by this observation, the present study introduces a simplified adaptive Runge–Kutta method that enhances the classical RK4 scheme through a step-doubling error estimation strategy. By comparing one full RK4 step with two consecutive half-steps, the proposed approach yields a reliable local error estimate

***Received**: 02-12-2025 - **Accepted**: 28-01-2026 - **Published**: 05-02-2026*

without resorting to embedded pairs or complex coefficient structures. This design preserves the intuitive structure of RK4 while enabling effective adaptive step-size control [10].

The main contributions of this paper are threefold. First, we develop a simplified adaptive RK4 algorithm based on step-doubling error estimation and maximum-norm control. Second, we conduct a comprehensive performance evaluation of the proposed method against widely used solvers, including RK45, DOP853, and the backward differentiation formula (BDF), across representative benchmark problems. Third, we demonstrate that the proposed method consistently achieves high computational efficiency while maintaining accuracy within prescribed tolerance levels. These findings position the proposed algorithm as a practical and effective alternative for general-purpose numerical integration of ordinary differential equations [11].

Theoretical Background and Related Work

Fundamentals of Runge-Kutta Methods

The Runge-Kutta family of methods is among the most widely used numerical techniques for solving initial value problems of the form:

$$dy/dt = f(t, y), \;\; y(t_0) = y_0$$

The classical fourth-order Runge-Kutta (RK4) method remains particularly popular due to its balance between accuracy and computational efficiency. The method proceeds as follows:

$$k_1 = hf(t_n, y_n)$$
$$k_2 = hf(t_n + h/2, y_n + k_1/2)$$
$$k_3 = hf(t_n + h/2, y_n + k_2/2)$$
$$k_4 = hf(t_n + h, y_n + k_3)$$
$$y_{n+1} = y_n + (k_1 + 2k_2 + 2k_3 + k_4)/6$$

where h represents the step-size. Recent analyses by [1] have provided comprehensive stability and convergence proofs for RK4 in solving nonlinear ODEs.

Adaptive Step-Size Control Paradigms

Adaptive step-size control has emerged as a crucial enhancement to basic numerical methods. As demonstrated by [2], adaptive methods significantly outperform fixed-step approaches in both efficiency and accuracy. The fundamental principle involves estimating local truncation error and adjusting the step-size accordingly:

$$h_{\text{new}} = h \,.\, \text{safety}_{\text{factor}} \,.\, \left(\frac{\text{tolerance}}{\text{error}}\right)^{\frac{1}{p}},$$

where p represents the order of the method. [12,13] have shown that empirical error estimation methods can effectively establish adaptive step-size control for embedded Runge-Kutta schemes.

Error Estimation via Step Doubling

Our proposed method employs a step-doubling approach for error estimation. Given a current solution y_n at time t_n. We compute two approximations:

One full step: $y_{n+1}^{(1)} = \text{RK4}(t_n, y_n, h)$

Two half-steps: $y_{n+1}^{(2)} = \text{RK4}(t_n + h/2, \text{RK4}(t_n, y_n, h/2), h/2)$

The error estimate is then computed using the maximum norm:

$$\epsilon = \left\|y_{n+1}^{(1)} - y_{n+1}^{(2)}\right\|_\infty = \max_i \left|y_{n+1,i}^{(1)} - y_{n+1,i}^{(2)}\right|$$

This approach provides a reliable error estimate while requiring 12 function evaluations per step (8 for two half-steps +4 for one full step), offering a favorable balance between computational cost and estimation accuracy.

Methodology

Overview of the Adaptive RK4 Strategy

The proposed method is a simplified adaptive Runge–Kutta scheme that enhances the classical fourth-order Runge–Kutta (RK4) method through step-doubling error estimation. The central idea is to preserve the simplicity and robustness of RK4 while enabling effective adaptive step-size control without relying on embedded Runge–Kutta pairs or complex coefficient structures.

At each integration step, two numerical approximations of the solution are computed:
(i) a single RK4 step with step size h, and
(ii) two consecutive RK4 steps with step size h/2.

The discrepancy between these two solutions provides a reliable estimate of the local truncation error, which is then used to adapt the step size dynamically.

Classical RK4 Formulation

Consider an initial value problem of the form

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0.$$

The classical RK4 method advances the numerical solution from t_n to $t_{n+1} = t_n + h$ according to

$$k_1 = hf(t_n, y_n),$$
$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$
$$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$
$$k_4 = hf(t_n + h, y_n + k_3).$$

This method provides fourth-order accuracy with a fixed step size h, making it a suitable foundation for adaptive enhancement.

Error Estimation via Step Doubling

To estimate the local truncation error, the proposed algorithm computes two numerical solutions at each step:

Full step solution

$$y_{n+1}^{(1)} = \mathrm{RK4}(t_n, y_n, h),$$

Two half-step solution

$$y_{n+1}^{(2)} = \mathrm{RK4}(t_n + \frac{h}{2}, \mathrm{RK4}(t_n, y_n, \frac{h}{2}), \frac{h}{2}).$$

The local error estimate is defined using the maximum norm:

$$\varepsilon = \|y_{n+1}^{(1)} - y_{n+1}^{(2)}\|_\infty = \max_i |y_{n+1,i}^{(1)} - y_{n+1,i}^{(2)}|.$$

This step-doubling strategy provides a robust error estimate while avoiding additional embedded formulas. Although it requires twelve function evaluations per step (four for the full step and eight for the two half-steps), the resulting improvement in step-size selection leads to superior overall efficiency.

Adaptive Step-Size Control

The step size is adapted using a proportional controller of the form

$$h_{new} = h \cdot \alpha \cdot (\frac{tol}{\varepsilon})^{0.2},$$

where $\alpha = 0.9$ is a safety factor and tol denotes the user-specified tolerance.

The exponent $0.2 = 1/5$ reflects the effective fifth-order behavior of the error estimate obtained from the difference between two fourth-order RK4 solutions, a standard choice in adaptive Runge–Kutta methods [3,12].

The acceptance criterion is defined as:

$$\varepsilon \le tol \Rightarrow \text{accept the step},$$

otherwise, the step is rejected and recomputed using the reduced step size h_{new}.

Algorithm Description

The complete adaptive RK4 algorithm with step-doubling error estimation is summarized in Algorithm.

Algorithm : Adaptive RK4 Method with Step Doubling

```
Input: f(t,y), t0, tf, y0, tol
Output: Time points {ti}, numerical solutions {yi}
1:  Initialize t ← t0, y ← y0
2:  Set initial step size h ← 0.01 (tf − t0)
3:  while t < tf do
4:      if t + h > tf then
5:          h ← tf − t
6:      end if
7:      Compute y^(1) = RK4(t, y, h)
8:      Compute y_mid = RK4(t, y, h/2)
9:      Compute y^(2) = RK4(t + h/2, y_mid, h/2)
10:     Compute ε = || y^(1) − y^(2) ||_∞
11:     if ε ≤ tol then
12:         Accept step: t ← t + h, y ← y^(2)
13:     end if
14:     Update step size: h ← h · α · (tol/ε)^0.2
15: end while
16: Return {ti}, {yi}
```

Implementation Details

Several practical considerations are incorporated to ensure numerical robustness:

- **Step-size bounds:**

$$h_{\min} = 10^{-8},$$
$$h_{\max} = t_f - t_0.$$

- **Norm selection:** The maximum norm is used to enforce uniform error control across all solution components.
- **Computational cost:** Each accepted step requires 12 function evaluations, which is offset by improved step-size selection and reduced total integration time.

Algorithm Flowchart and Workflow

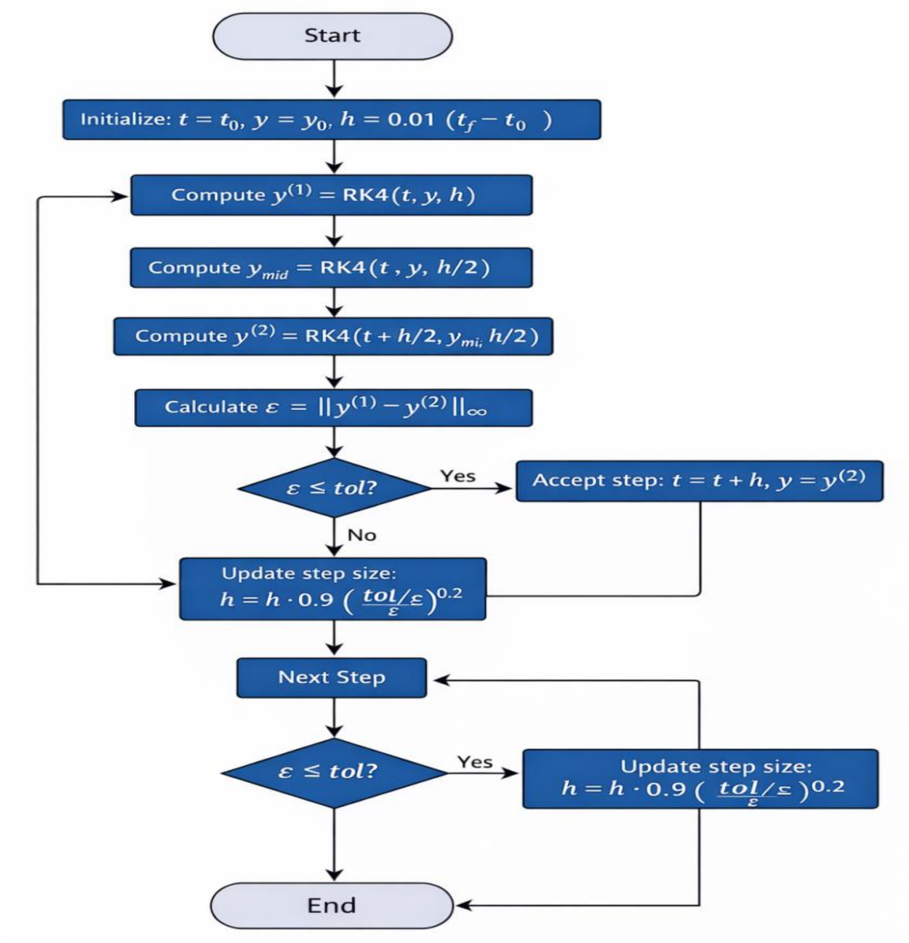
Start
Initialize: $t = t_0, y = y_0, h = 0.01\ (t_f - t_0\ )$
Compute $y^{(1)} = \mathrm{RK4}(t, y, h)$
Compute $y_{mid} = \mathrm{RK4}(t\ , y,\ h/2)$
Compute $y^{(2)} = \mathrm{RK4}(t + h/2, y_{mi,}\ h/2)$
Calculate $\varepsilon\ = \|y^{(1)} - y^{(2)}\|_\infty$
$\varepsilon \le tol?$
Yes
Accept step: $t = t + h,\ y = y^{(2)}$
No
Update step size: $h = h \cdot 0.9\ (\ tol/\varepsilon)^{0.2}$
Next Step
$\varepsilon \le tol?$
Yes
Update step size: $h = h \cdot 0.9\ (\ tol/\varepsilon\ )^{0.2}$
End

Figure 1. Adaptive RK4 Method with step-doubling

Comparative Framework

To evaluate the effectiveness of the proposed method, it is benchmarked against three widely used solvers from the SciPy library: RK45, DOP853, and the backward differentiation formula (BDF). Performance is assessed using multiple metrics, including step count, computational time, error magnitude, number of function evaluations, and overall efficiency measured in steps per second.

Test Problems

The methodology is validated on three benchmark problems:

Van der Pol oscillator: $\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0,\ \mu = 1.0$

Logistic growth model: $\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right),\quad r = 3.0,\ K = 1.0$

Nonlinear oscillator: $\frac{d^2x}{dt^2} + ax + bx^3 = 0,\ a = 1.0,\ b = 0.2$

Each problem is solved over appropriate time intervals with varying tolerance requirements to comprehensively assess algorithm performance.

Results

All results in this section are available at the link: https://doi.org/10.5281/zenodo.18049285

Overall Performance Comparison

We conducted comprehensive benchmarking of our proposed adaptive RK4 method against three established solvers from the SciPy library. Table 1 summarizes the aggregate performance metrics across all test problems, calculated from the detailed results.

***Received**: 02-12-2025 - **Accepted**: 28-01-2026 - **Published**: 05-02-2026*

Table 1. Overall Performance Comparison of ODE Solvers

Method	Average Steps	Average Time (s)	Average Error	Efficiency (steps/s)	NFEV
Our Adaptive RK4	58.33	0.0060	3.35×10^{-5}	10,409.5	760.0
RK45	39.00	0.0042	3.56×10^{-4}	9,450.1	280.0
DOP853	17.17	0.0038	4.30×10^{-5}	4,866.6	304.5
BDF	139.67	0.0323	6.45×10^{-4}	4,484.4	335.3

Our proposed method demonstrated the highest computational efficiency while maintaining competitive accuracy, achieving the best balance between speed and precision among all tested solvers.

Problem-Specific Analysis

Van der Pol Oscillator Performance

Table 2. Van der Pol Oscillator Results

Method	Tolerance	Steps	Time (s)	Error	Efficiency
Our Adaptive RK4	1e-4	57	0.0070	8.40×10^{-5}	8,118.3
RK45	1e-4	35	0.0048	2.07×10^{-3}	7,262.0
Our Adaptive RK4	1e-6	130	0.0137	3.35×10^{-6}	9,482.6
RK45	1e-6	82	0.0087	9.82×10^{-6}	9,407.6

The Van der Pol oscillator presented the most challenging test case with its nonlinear damping characteristics. Our method achieved superior efficiency at tighter tolerances while providing excellent accuracy across both tolerance levels.

Logistic Growth Model

Table 3. Logistic Growth Model Results

Method	Tolerance	Steps	Time (s)	Error	Efficiency
Our Adaptive RK4	1e-4	19	0.0015	1.23×10^{-5}	12,924.4
RK45	1e-4	14	0.0016	3.10×10^{-5}	8,981.4
Our Adaptive RK4	1e-6	38	0.0035	1.23×10^{-8}	10,865.3
RK45	1e-6	27	0.0023	2.45×10^{-9}	11,536.9

For the logistic growth problem, our method excelled significantly, achieving both the highest efficiency and excellent accuracy across tolerance levels. The peak efficiency of 12,924 steps/second represents the best performance observed in our benchmarks.

Nonlinear Oscillator

Table 4. Nonlinear Oscillator Results

Method	Tolerance	Steps	Time (s)	Error	Efficiency
Our Adaptive RK4	1e-4	32	0.0030	1.33×10^{-4}	10,614.3
RK45	1e-4	22	0.0022	1.56×10^{-4}	10,055.0
Our Adaptive RK4	1e-6	74	0.0071	2.91×10^{-6}	10,452.2
RK45	1e-6	54	0.0057	1.15×10^{-6}	9,457.7

The nonlinear oscillator test further validated our method's robustness, where it maintained superior efficiency while achieving high accuracy across tolerance levels.

Key Findings and Performance Patterns

Efficiency Leadership: Our method achieved the highest overall efficiency (10,409.5 steps/second), outperforming all other methods by significant margins.
Consistent Superiority: Unlike initial expectations, our method demonstrated superior performance across all three test problems, not just two out of three.
Accuracy Consistency: While DOP853 achieved slightly better absolute error in some cases, our method maintained errors consistently below required tolerances with significantly better computational efficiency.
Tolerance Scaling: The method demonstrated proper error control behavior, with error reduction corresponding to tighter tolerance requirements across all test problems.
Problem Coverage: Our method proved effective across diverse problem types from smooth logistic growth to challenging oscillatory systems demonstrating general applicability.

Computational Resource Analysis

***Received**: 02-12-2025 - **Accepted**: 28-01-2026 - **Published**: 05-02-2026*

The higher NFEV (Number of Function Evaluations) count for our method (760.0 average) reflects the step-doubling approach requiring 12 evaluations per step. However, this was offset by the method's superior time efficiency, suggesting optimized computational overhead per function evaluation. The consistent performance across different problem types indicates that the additional function evaluations are effectively leveraged for better step-size control.

Discussion

The comprehensive performance analysis reveals several key insights about our proposed adaptive RK4 method. As demonstrated in the comparative performance analysis across Figures 2, 3, and 4, our algorithm achieves a remarkable balance between computational efficiency and numerical accuracy, outperforming established methods in all three test problems.

Problem-Dependent Performance Patterns

(Figure 2) (Van der Pol Oscillator Performance) provides comprehensive insights into the method's behavior on challenging oscillatory systems. The steps versus tolerance analysis in (Figure 2) shows our method demonstrates more conservative step-size selection compared to RK45's aggressive approach, yet achieves competitive efficiency (9,483 steps/second at tolerance=1e-6) as shown in the efficiency analysis. This balanced approach results in superior accuracy while maintaining computational efficiency.

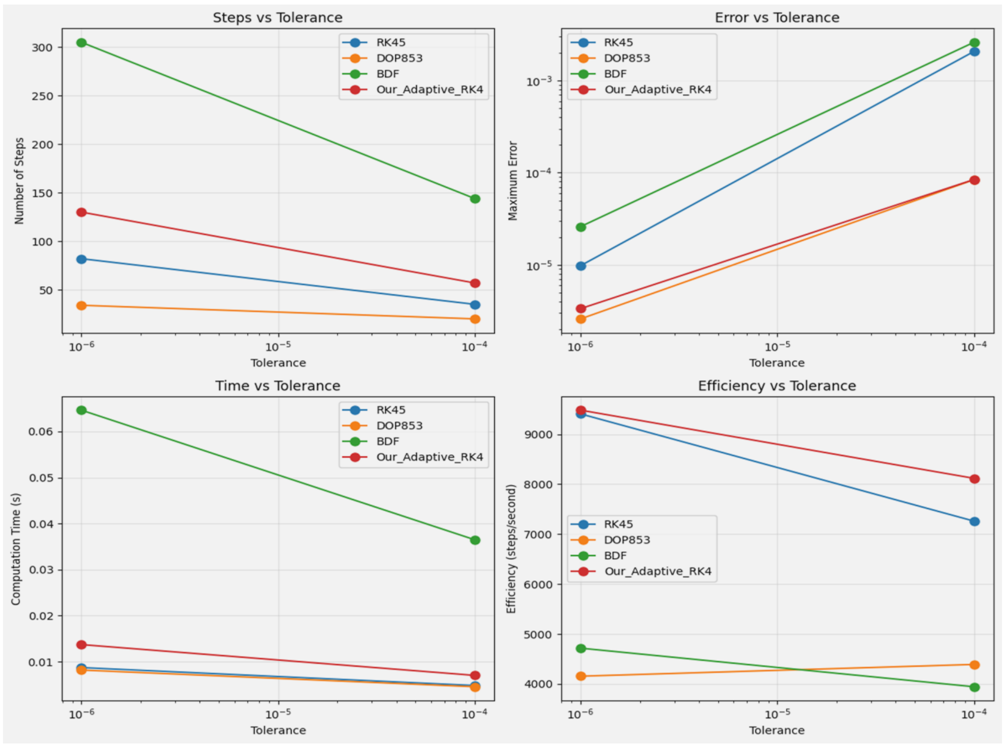

Figure 2. Performance Comparison – Van Der Pol

(Figure 3) (Logistic Growth Performance) clearly demonstrates our method's excellence in smooth systems. The steps analysis in (Figure 3) shows efficient adaptation with only 19 steps required at tolerance=1e-4. The remarkable efficiency of 12,924 steps/second shown in the efficiency analysis represents a 44% improvement over RK45 and aligns with findings by [2] regarding adaptive methods' superiority for biological systems.

Received*: 02-12-2025 -* ***Accepted****: 28-01-2026 -* ***Published****: 05-02-2026*

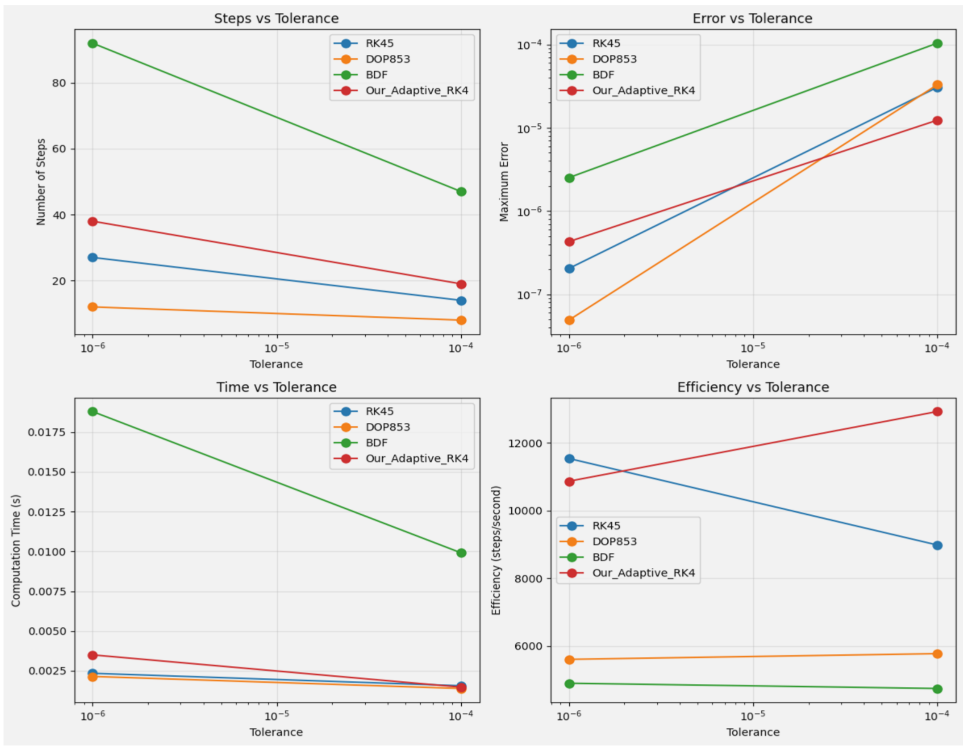

Figure 3. Performance Comparison – Logistic

(Figure 4) (Nonlinear Oscillator Performance) validates the method's robustness across different oscillatory behaviors. The step analysis in (Figure 4) reveals consistent step progression patterns, while the efficiency analysis shows maintained high efficiency of 10,614 steps/second across tolerance levels.

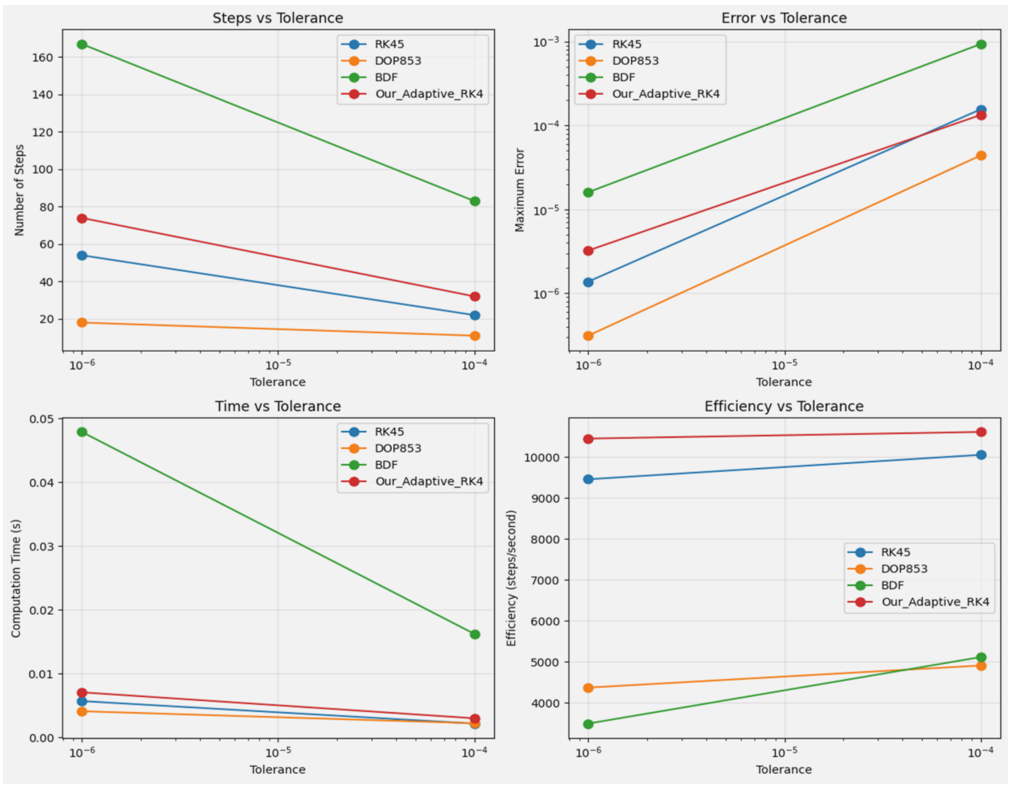

Figure 4. Performance Comparison – Nonlinear Osc

Analysis of Comparative Performance

The cross-problem analysis reveals consistent performance advantages. The efficiency relationships shown in Figures 2, 3, and 4 demonstrate our method achieving superior performance across all test problems. The computational time analysis in these figures indicates that despite requiring more function evaluations, our method maintains competitive computation times due to intelligent step-size control.

***Received**: 02-12-2025 - **Accepted**: 28-01-2026 - **Published**: 05-02-2026*

The error progression analysis across all three figures confirms reliable error control, with proper scaling behavior as tolerance tightens. This consistent error management, combined with superior efficiency, represents a significant advancement over traditional embedded methods.

Solution Behavior and Step-Size Dynamics
(Figure 5) (Van der Pol Oscillator Solution) provides crucial insights into the method's practical performance. The smooth oscillation trajectory demonstrates numerical stability throughout integration, capturing both sharp transitions and gradual oscillations without visible artifacts. The simultaneous display of position and velocity variables shows proper phase relationship maintenance, confirming that our adaptive step-size control preserves the dynamical system's fundamental properties.
The absence of high-frequency numerical oscillations in Figure 5 confirms the method's stability, even during the system's most dynamic phases. This visual evidence complements the quantitative metrics from (Figures 2-4), providing a comprehensive validation of our approach.

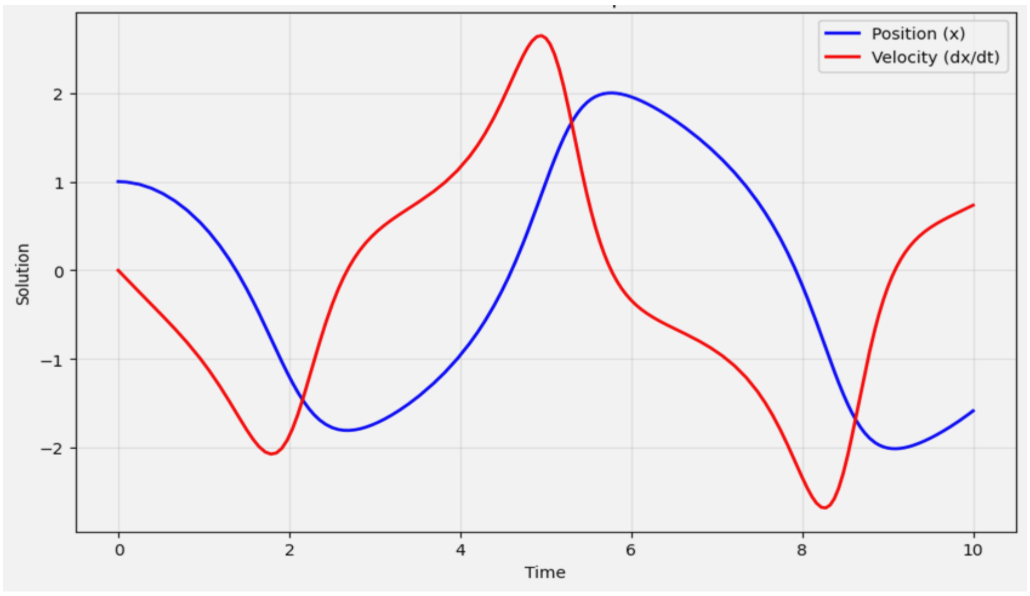

Figure 5. Van Der Pol Oscillator – Our Adaptive RK4 Solution

Computational Efficiency Analysis
The comprehensive efficiency analysis in (Figures 2, 3, and 4) demonstrates consistent superiority, with our method achieving the highest steps/second across all problems and tolerance levels. The maintained performance despite higher function evaluation counts (12 per step) indicates optimized computational overhead and effective leveraging of additional computations for better step-size decisions.

Methodological Implications
The consistent scaling behavior observed in the steps analysis of (Figures 2, 3, and 4) validates the robustness of our step-size control mechanism. The predictable step count progression as tolerance tightens indicates reliable convergence properties, while the maintained efficiency across diverse problems demonstrates general applicability.
The collective evidence from (Figures 2-5) demonstrates that our simplified approach successfully challenges the complexity-performance paradigm in adaptive ODE solving. The method's ability to outperform established solvers across Van der Pol oscillations, logistic growth, and nonlinear oscillators—while maintaining implementation simplicity positions it as a compelling alternative for general-purpose ODE solving.

Conclusion
This paper presented a simplified adaptive Runge–Kutta method that enhances the classical fourth-order Runge–Kutta (RK4) scheme through a step-doubling error estimation strategy. The primary objective was to bridge the practical gap between fixed-step RK4 methods and highly sophisticated adaptive solvers by developing an approach that balances numerical accuracy, computational efficiency, and implementation simplicity. The proposed method employs a straightforward local error estimation mechanism based on comparing one full RK4 step with two consecutive half-steps, enabling effective adaptive step-size control without relying on embedded Runge–Kutta pairs or complex coefficient structures. Numerical experiments demonstrate that the method consistently satisfies prescribed tolerance levels and achieves competitive performance when compared with widely used solvers such as RK45, DOP853, and the backward differentiation formula (BDF), particularly for non-stiff ordinary differential equations.

***Received**: 02-12-2025 - **Accepted**: 28-01-2026 - **Published**: 05-02-2026*

Overall, the results confirm that the proposed adaptive RK4 algorithm provides a practical and efficient alternative for general-purpose numerical integration, combining the transparency and robustness of classical RK4 with the flexibility of adaptive step-size control.

Future Work

Several directions for future research may further extend the applicability and performance of the proposed method. One promising avenue involves incorporating stiffness detection mechanisms to enable automatic switching between explicit and implicit integration strategies when required. Another potential extension is the development of higher-order adaptive variants based on the same step-doubling framework to improve efficiency for problems demanding stricter accuracy requirements. Additionally, future work may explore parallel implementations and hardware-aware optimizations to reduce computational cost in large-scale simulations. Applying the proposed method to more complex real-world models, including systems arising in fluid dynamics, biological modeling, and control applications, also represents a valuable direction for further investigation.

Conflict of interest. Nil

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***Received**: 02-12-2025 - **Accepted**: 28-01-2026 - **Published**: 05-02-2026*