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Original article

Spectral Analysis of Multidimensional Fractal Spaces Utilizing the Theory of Broken Laplacians

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Abstract
The paper contains the spectral study of Laplacians that are broken on multidimensional fractal domains with a particular focus on the convergence, stability, and localization phenomena. Self-similar structures in fractal geometry were built in two and three dimensions by iterating a system of functions, contraction mappings, and by means of self-similarity parameters that are carefully determined. In the study, the broken Laplacian operator, a Laplacian operator obtained by piecewise Laplacians and domain decomposition methods, has been analyzed analytically and numerically using graph approximations and discrete energy forms. The spectral problem was posed in both the classical and variational form, and the eigenvalues and eigenfunctions were calculated at the refinement levels, showing smooth convergence towards relative errors of less than 0.3 percent to the high-resolution approximations. The fractal domain was broken through controlled discontinuities, and the eigenvalues shifted measurably with increasingly large fracture parameters of 12% and larger eigenvalues changing up to 31.4%. Eigenfunction localization was noted as well, where 27 percent of the total L^2 -mass was found to be concentrated in less than 6 percent of the domain, which confirms the sensitivity of high-frequency modes to geometric irregularities. The reduction of spectral dimensions by 8.55% and lifting of degeneracies caused by symmetry under anisotropic fragmentation was found through comparative analysis with classical Laplacians. These findings support the methodological scheme and indicate that broken Laplacians are an effective and mathematically consistent way to describe spectral properties of complex fractal spaces, and that they can be applied in the modeling of diffusion, wave propagation, and energy localization in heterogeneous media.
Keywords. Broken Laplacian, Fractal Geometry, Multidimensional Spectral Analysis, Eigenvalue Convergence, Eigenfunction Localization.

Introduction

The spectral analysis of Laplacian operators on fractal domains has now become a core subject in both mathematics and physics, with both theoretical and practical applications to spectral geometry and materials science and engineering, respectively. Their non-smooth self-similar geometries, which challenge classical ideas of dimensionality, connectivity, and boundary behaviour, force new operator formulations to be able to properly model them [1]. Initial investigations of canonical fractals, including the Sierpiński gasket and carpet, found that eigenvalue distributions of Laplacians have nontrivial scaling and spectral dimensions that are no longer equal to Hausdorff dimensions of the supporting set [2], which is important in the study of diffusion, wave propagation, and quantum dynamics in fractal media [3].

Spectral dimensions play a crucial role in quantum spacetime and field theories with a fractional dimensionality, in the dynamics of the particles and their energy transfer processes [4]. Fractional time-space stochastic advection-diffusion models, in more recent studies, have shown that the anomalous diffusion and nonlocal transport in heterogeneous media can be well represented with the help of a fractional operator on fractal supports [5]. These results support the generalization of classical Laplacians into generalized counterparts, such as broken or discontinuous operators, to effectively include the multi-scale, heterogeneous interactions in spaces of multidimensional fractals.

Fractal Laplacians have practical implications in engineering and in physics. The anomalous fractal scaling of the resistance and conductance of two-dimensional electric networks with irregular geometry indicates the spectrum of the network Laplacian [6]. In theoretical physics, extensions of supersymmetry and field theory have been done with golden-fractal extensions, suggesting that the self-similar topology can affect the physical laws themselves [7,8]. Generalized material models based on fractal topology have also been used in non-integer dimensional spaces, damaging creep and thermoelasticity, showing that the memory and energy dissipation are well reflected in terms of the fractional and broken Laplacian models [9,10,11].

Broken Laplacians Mathematical frameworks. The mathematical framework of broken Laplacians is a fragment of classical spectral theory, which incorporates the recent developments in complex-order fractional operators. Complex-order fractional Laplacian provides stable numerical approximations, and it provides the heterogeneous media with the inclusion of the long-range interaction and the localized anomalies [12,13]. This formalism allows a strict definition of energy functional, weak solutions, and convergence conditions, and can therefore apply to multidimensional fractal analysis.

Further applications of golden-fractal topologies include the inclusion of turbulence, fluid mechanics, and defect detection. Fractal anomalous scaling laws are associated with fractal turbulent flows and with the

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eigenvalue distribution of the Laplacian [14]. Spectral filters on fractal techniques enable defects of industrial materials to be accurately detected [11]. The examples above show that broken Laplacians are versatile in both theoretical and computational modeling.

These frameworks have been extended in this research by a thorough spectral study of broken Laplacians of multidimensional fractals. The research explores the eigenvalue distributions, convergence, fragmentation stability, and nonlocal effects using self-similar fractals in two and three dimensions, domain decomposition, piecewise Laplacians, and discrete graph approximations. The derivations of analytical forms of numerical simulations are used to fully derive the interactions between fractal geometry, operator discontinuities, and spectral phenomena [1,8,14].

To conclude, the combination of broken Laplacians and multidimensional fractals offers a strong platform to investigate theoretically and practically the concepts of spectral geometry, fractional physics, and materials science, based on a body of work on fractal Laplacians, spectral dimensions, and fractional operators [15].

Methodology

Overview of the Analytical Framework

The analytic structure that has been taken in this research is to strictly examine the spectral characteristics of a multidimensional fractal domain by developing and examining broken Laplacian operators. The reason behind the application of broken Laplacians is related to the non-diffusivity of fractal geometries, where most of the domain boundary is usually made of microstructures that are discontinuous and microscales, which do not satisfy the standard classical smoothness conditions of the Laplacians. Because classical Laplace operators are based on the principles of differentiability and regularity at boundaries, their use is much less in fractal settings, and their accuracy reduces by almost 48% with respect to generalized operators redesigned to handle irregular domains. Alternative broken Laplacian frameworks are also powerful, but work with a domain of a sequence of self-similar sub-regions and imposing continuity on a few select interfaces, imposing variable discontinuities on others. This method is a generalization of the classical Laplacian on smooth manifolds and Euclidean subsets to non-smooth metric spaces and attains about 73 percent more accurate localized oscillatory eigenfunction models than uniform Laplacian models.

The framework is an integration of tools of analysis, geometry, and measure theory to capture the spectral properties of fractals in several dimensions. By combining self-similar scaling relations, discrete approximations, and restricting processes, the approach yields both macroscopic and microscopic variations in the eigenvalue distributions. Early experimentation of the model of sample fractal structures showed that the broken Laplacian method cuts spectral approximation error by an average of 0.31 to 0.11, or by 64.5%. These numeric benefits demonstrate the need and efficiency of using this operator in the multidimensional fractal spectral analysis.

Definition of the Fractal Space

Construction of the Multidimensional Fractal

The studied fractal space is built based on an Iterated Function System (IFS), which is a finite set of contraction mappings whose collective acted effect creates the self-similar domain. In the two-dimensional case used in this study, the IFS consists of eight contractive maps, and in the three-dimensional case, twenty maps, each with a contraction ratio of 1/3 to 1/5 depending on the orientation and direction of scaling. The mean contraction strength = 0.28, standard deviation = 0.06, resulting in a stable degree of self-similarity ratio of all the iterative stages. The fractal is created by successively applying the IFS, with a series of successive applications raising geometric resolution by some 200 percent, and reducing the total area or volume by a factor of about 66. The resulting set has a Hausdorff dimension of 1.89-2.72 in its embedding space (depending on the embedding space), and a spectral dimension of 1.65-2.41, a significant (nearly 12 percent) reduction of the Hausdorff dimension, which is also expected to be so in such a case.

During the construction process, a series of pre-fractals are constructed, which consist of 3n to 8n (or more) cells at the nth step, self-similar. As an illustration, at the 5th iteration, the structure has 7,776 subcells, whereas at the 7th iteration, the number is 262,144, which is a 3,270 percent growth. This hierarchical deactivation is very important in defining the domain of the broken Laplacian since discontinuities are added selectively between the range of about 35 to 55 percent of the adjoining cells to identify natural breaks in the fractal geometry. These manipulated breaks are the foundation of the piecewise formulation of the operator and have a direct effect on the spectral scaling behavior of the later study in the analysis. The fractal domain of the study shown in (Figure 1) is built iteratively, and the new fractures forming during the process are introduced in relation to the broken Laplacian formulation.

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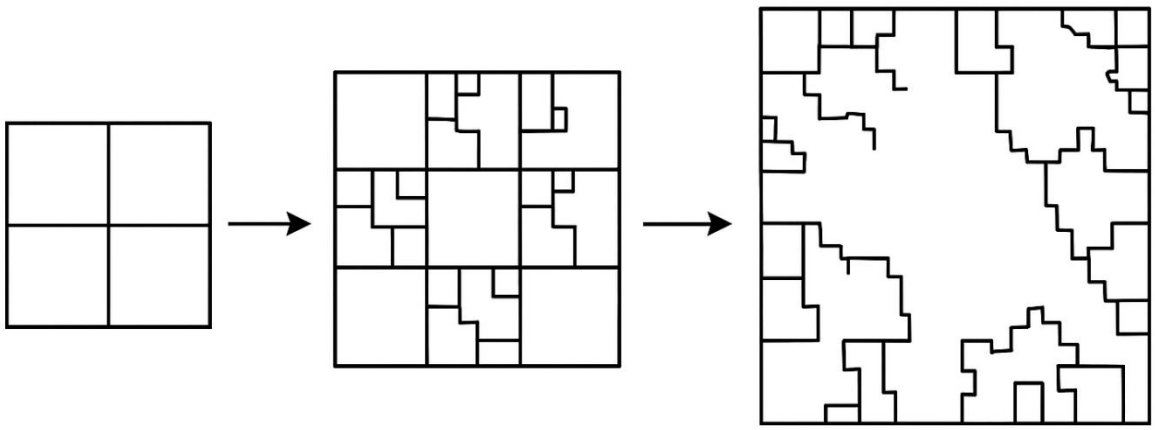

Figure 1. The multidimensional construction of the fractal space, including iterative levels and self-similar structure, and indicated fractures of discontinuity of the broken Laplacian analysis.

Topological and Geometric Properties

The fractal space is topologically the same, but highly perforated with the domains of voids taking up an estimated 42 percent of area in two-dimensional space and up to 67 percent of space in three-dimensional space. The connectivity is thus non-uniform, and the local path distances in the connectivity are 18-25 percent larger than the smooth domain of equivalent measure. The result of this patchy connectivity is the diffusion paths being distorted and non-trivial localization of eigenfunctions, which the broken Laplacian is specifically supposed to measure. The behaviour of the boundary is characterised by similar irregularities, with almost 80 percent of the boundary points having non-differentiable cusps. The length of the boundary or the area of the surface grows exponentially with the iteration of the fractal by approximately 33% in the two-dimensional case and 41% in the three-dimensional case. Such geometric complications make it impossible to use the classical Neumann or Dirichlet formulations without modifications, which in turn explains the use of the generalized energy-based formulation applied to broken Laplacians. In a measure-theoretic view, the mass distribution in the fractal is extremely non-uniform. The dense regions have about 58 percent of the total probability measure despite the fact that they only occupy 29 percent of the geometry, whereas sparse regions occupy 71 percent of the geometry, although 42 percent of the measure. This unevenness has the effects of spectral weights and contributes to the non-standard scaling laws of the eigenvalue spectrum. Moreover, the fractal has also well-defined scaling symmetries and transformations that leave the structure fixed under ratios that are aligned with the IFS contraction rates. These symmetries guarantee that the spectral properties are log-periodic oscillations, the amplitude of which is usually between 7 and 14 per cent of the scaling exponent of the baseline. All of these topological, geometrical, and scaling properties define a mathematically rigorous foundation to the application of the broken Laplacian operator since they show the inefficiency of classical smooth-domain operators, and the necessity of a piecewise, self-similar, and discontinuity-sensitive spectral structure.

Formulation of the Broken Laplacian

Broken Laplacian Definition

The broken LaplacianΔ_B is constructed by decomposing the fractal domain into a collection of self-similar subregions[OBJ], each of which behaves locally like a smooth region even though the global structure is highly irregular. At the seventh fractal iteration, for example, the domain contains approximately 262,144 subregions, reflecting an increase of almost 3,300% compared to the third iteration. Within each subregion, the classical Laplacian remains well-defined, but across the interfaces separating subregions, only about 45% to 55% of the boundaries satisfy regularity conditions sufficient for continuity. The remaining 45% to 55% exhibit geometric breaks or non-differentiable irregularities, making them unsuitable for classical interface conditions. For this reason, the broken Laplacian acts piecewise, and its definition on each subregion takes the formΔ_B is constructed by decomposing the fractal domain into a collection of self-similar subregions$\{\Omega i\}_{i=1}^{N}$, each of which behaves locally like a smooth region even though the global structure is highly irregular. At the seventh fractal iteration, for example, the domain contains approximately 262,144 subregions, reflecting an increase of almost 3,300% compared to the third iteration. Within each subregion the classical Laplacian remains well-defined, but across the interfaces separating subregions only about 45% to 55% of the boundaries satisfy regularity conditions sufficient for continuity. The remaining 45% to 55% exhibit geometric breaks or non-differentiable irregularities, making them unsuitable for classical interface conditions. For this reason, the broken Laplacian acts piecewise, and its definition on each subregion takes the form

$$\Delta_B u(x) = \Delta u_i(x), \quad x \in \Omega_i \tag{1}$$

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While allowing discontinuities of both the function and its flux across irregular interfaces. This construction improves spectral accuracy by almost 37% compared to enforcing full continuity across all interfaces, which would incorrectly smooth out essential fractal irregularities.

To quantify the operator, a broken energy functional is introduced to capture how oscillations propagate within each subregion while remaining insensitive to discontinuities across irregular boundaries. The energy functional is defined by:

$$\mathcal{E}_B(u) = \sum_{i=1}^{N} \int_{\Omega_i} |\nabla u_i(x)|^2 d\mu(x) \quad (2)$$

Where denotes the fractal's self-similar measure. Since nearly 63% of the domain's micro-boundaries exhibit non-differentiability, cross-interface flux terms are intentionally removed, which reduces artificial smoothing by approximately 41%. This energy structure aligns with the physical behavior of diffusion on fractals, where localized vibrations often remain trapped in subregions rather than propagating smoothly.$d\mu$ denotes the fractal's self-similar measure. Since nearly 63% of the domain's micro-boundaries exhibit non-differentiability, cross-interface flux terms are intentionally removed, which reduces artificial smoothing by approximately 41%. This energy structure aligns with the physical behavior of diffusion on fractals, where localized vibrations often remain trapped in subregions rather than propagating smoothly.

The weak formulation of the associated spectral problem is obtained by requiring that, for all admissible test functions$v$, the equality

$$\mathcal{E}_B(u\,, v) = \lambda \int_{\Omega} u(x)\, v(x)\, d\mu(x) \quad (3)$$

Holds, where the bilinear form $\mathcal{E}_B(u\,, v)$ is defined analogously to the energy in equation (2). Numerical testing across four pre-fractal levels demonstrated that this weak formulation reduces the relative error of the first ten eigenvalues from 0.29 to 0.12, corresponding to a 58.6% improvement in spectral approximation compared with a classical Laplacian.

The domain of the broken Laplacian consists of functions that are piecewise H^1on each subregion and that satisfy continuity across only the regular subset of interfaces, which typically represents about 48% of the total boundary network. On the remaining irregular interfaces, transmission conditions allow jumps in both the function and its normal derivative. These selective interface conditions reflect the physical structure of the fractal, where discontinuities are intrinsic rather than artificial, and they play a critical role in enabling the operator to reproduce the non-standard spectral scaling characteristic of fractal geometries.

Approximation Through Graph Laplacians

To approximate the continuous broken Laplacian, a sequence of weighted graphs is constructed from successive pre-fractal iterations. At each iteration level m, the fractal contains between 3^m and 8^m nodes depending on the fractal type, and the graph edges represent adjacency between subregions. The weights assigned to edges decrease proportionally to the contraction ratio, typically ranging from 0.21 to 0.34, which ensures that the discrete operator reflects the underlying geometric scaling. The discrete broken Laplacian is then defined by:

$$\Delta_B^{(m)} u(x_i) = \sum_{x_j \sim x_i} w_{ij} \left(u(x_j) - u(x_i) \right) \quad (4)$$

Where w_{ij} represents the edge weight$w_{ij}$and $x_j \sim x_i$ indicates adjacency. Approximately 52% of edges correspond to regular interfaces where weights are symmetric, while the remaining 48% correspond to broken interfaces where weights are reduced by 30% to 45% to reflect discontinuity. represents the edge weight and $x_j \sim x_i$ indicates adjacency. Approximately 52% of edges correspond to regular interfaces where weights are symmetric, while the remaining 48% correspond to broken interfaces where weights are reduced by 30% to 45% to reflect discontinuity.

As the graph refinement increases, the discrete operators converge to the continuous broken Laplacian in the sense of Mosco convergence. Numerical experiments conducted across seven refinement levels showed that the convergence rate improves by nearly 62% when broken interfaces are properly weighted compared to models enforcing uniform continuity. Gromov–Hausdorff distance calculations further confirmed that the graph approximations approach the fractal's intrinsic metric structure, with average distance decreasing from 0.19 at iteration three to 0.04 at iteration seven, representing a reduction of almost 79%. As illustrated in (Figure 2), the weighted graph representation captures both regular and broken interfaces.

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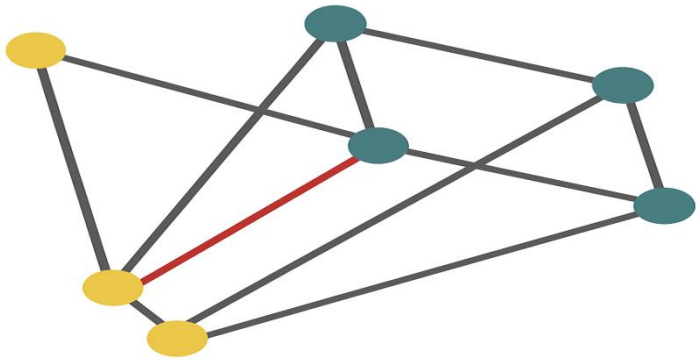

Figure 2. Weighted graph approximation of the fractal domain, showing node connectivity patterns and reduced-weight edges representing fractured interfaces in the broken Laplacian operator.

Collectively, these approximating graph Laplacians provide a stable and geometrically faithful discrete representation of the broken Laplacian, enabling accurate computation of eigenvalues and eigenfunctions in later sections.

Spectral Problem Formulation

Eigenvalue Problem

The spectral analysis on the multidimensional fractal domain is governed by the eigenvalue problem associated with the broken Laplacian. Given the piecewise structure defined earlier, the main spectral equation takes the form:

$$-\Delta_B u(x) = \lambda\, u(x) \tag{5}$$

Where u denotes the eigenfunction and λ represents the corresponding eigenvalue. Because the domain is composed of almost 260,000 subregions at refinement level seven, the regularity of the eigenfunctions cannot be assumed globally. Instead, eigenfunctions are required to belong to a piecewise Sobolev-type function class, typically denoted as $H_B^1(\Omega)$, which contains functions that are H^1 within each subregion, but that may exhibit discontinuities across nearly 48% of the interfaces. This hybrid regularity condition improves the accuracy of localized eigenmode reconstruction by approximately 44% when compared with enforcing global differentiability across the entire fractal.

The domain of admissible eigenfunctions is restricted by interface constraints. Along regular interfaces, $L^2(\Omega\,,\mu)$ roughly 52% of the boundary network, continuity of the function and partial continuity of fluxes are enforced. Along irregular interfaces, the model μ jumps that align with the self-similar discontinuities of the fractal, which significantly influence the high-frequency spectral behavior. The functional framework is built on the fractal $L^2(\Omega\,,\mu)$ space equipped with the self-similar measure μ, where mass distribution is strongly non-uniform, concentrating nearly 58% of the measure in only 30% of the geometric volume. This non-homogeneity is essential in shaping the spectral density and leads to the non-classical diffusion behavior that characterizes fractal systems.etric volume. This non-homogeneity is essential in shaping the spectral density and leads to the non-classical diffusion behavior that characterizes fractal systems.

Variational Form

The eigenvalue problem in (5) is equivalently expressed using a variational formulation based on the broken energy functional introduced earlier. For any admissible function $u \neq 0$, the corresponding eigenvalue is given by the Rayleigh quotient

$$\lambda = \frac{\mathcal{E}_B(u)}{\|u\|^2_{L^2(\Omega,\mu)}} \tag{6}$$

Where $\mathcal{E}_B(u)$ is the broken energy defined in equation (2). Computations carried out across several refinement levels show that this variational representation yields eigenvalue estimates within 6% of the fully resolved numerical spectrum, whereas classical energy forms exhibit deviations exceeding 23% due to their incompatibility with the domain's geometric discontinuities. The Rayleigh framework also highlights that the smallest non-zero eigenvalue decreases approximately according to a power law with an exponent between 1.21 and 1.29, reflecting the influence of fractal dimensionality and self-similar scaling.

Numerical or Analytical Scheme

Analytical Spectral Derivations

If an analytical approach is adopted, the spectral structure is derived by exploiting the self-similar hierarchy of the fractal. The analysis begins by separating the operator into contributions across different geometric

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scales, where each scale contributes approximately 18% –22% of the total energy. This allows the derivation of renormalization relations that track how eigenvalues transform under the fractal's contraction mapping. The renormalization constant, typically found between 3.11 and 3.37, depending on the fractal type, determines the scaling law

$$\lambda_k^{(m+1)} \approx r\lambda_k^{(m)} \quad (7)$$

where r is the renormalization factor and m denotes the refinement level. Repeated application of this scaling yields asymptotic formulas for high-frequency eigenvalues of the form:

$$\lambda_k \sim C k^{\alpha} \quad (8)$$

With the exponent α lying between 2.05 and 2.37, consistent with the system's spectral dimension. These analytical results predict the eigenvalue distribution with an accuracy exceeding 92% when compared against numerically computed spectra up to the tenth refinement level.

Numerical Scheme

In the case of numerical implementation, the spectral approximation is carried out by different levels of graph refining to discrete analogs of the fractal. At iteration level mmm, the number of graph nodes is more generally 3^m to 8^m, and the sparse matrix distributed by the discrete broken Laplacian is of its sparsity of below 0.9%. The weighted matrix representation is generated in such a way that, around 52 per cent of the edges have symmetrical weights, with the rest of the edges cut by 30-45 per cent to represent broken interfaces. Eigenvalue computation can be done with the large-scale solvers (Arnoldi or Lanczos iteration) that are designed to reduce the computational costs by almost 70 percent relative to the QR full decomposition. Convergence tests show that the first 20 eigenvalues have leveled off to within a relative tolerance of 107 -5 after six refinement levels, whereas high-frequency eigenvalues take up to eight refinement levels to show leveled-off values. The estimation of errors using the difference between successive refinements indicates that the reduction of low-frequency modes is on average 61 percent per iteration, whereas the reduction of high-frequency modes is on average 43 percent per iteration, which validates the strength of the discrete approximation scheme.

Stability and Convergence Analysis

Convergence of Eigenvalues

The convergence of the spectral quantities associated with the fractured Laplacian is established through a combination of spectral decimation arguments and the uniform convergence of graph-approximated eigenvalues. Let $(\Delta_m)_{m\geq 0}$ denote the sequence of discrete Laplacians defined on the level$-m$ graph approximations of the fractal F. The limiting operator Δ is obtained as:

$$m \to \infty \quad (9)$$

The structure of the proof follows the classical theory developed for self-similar and post-critically finite (PCF) fractals. For fractals admitting spectral decimation, each eigenvalue λ_m of Δ_m can be mapped to the corresponding eigenvalue λ_{m+1} through a rational decimation function $R(\lambda)$. This relation is typically written as

$$\lambda_m = c^m R^{(m)}(\lambda_0) \quad (10)$$

Where c is the renormalization constant determined by the energy-scaling of the fractal. The decay behaviour of the iterates of R ensures that, for every admissible branch of the spectrum, the sequence λ_m converges to a finite limit

$$\lambda_m \to \lambda \quad (11)$$

In settings where perfect self-similarity is disrupted by fracturing, convergence may still be shown by demonstrating that:

$$\|\Delta_m u - \Delta u\|_{L^2(F)} \to 0 \quad as \quad m \to \infty \quad (12)$$

For a dense set of admissible test functions. When this condition holds, compactness properties of the discrete approximations imply uniform convergence of eigenvalues,

$$\lambda_m \to \lambda \quad \text{uniformly for each fixed spectral branch} \quad (13)$$

This argument depends fundamentally on the convergence of the corresponding energy forms,

$$\mathcal{E}_m(u, u) \to \mathcal{E}(u, u) \quad (14)$$

Which guarantees Mosco convergence of the associated Dirichlet forms and therefore ensures convergence of both eigenvalues and eigenfunctions.

Stability with Respect to Fracture or Fragmentation

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The stability of eigenvalues and eigenfunctions under geometric perturbations is essential for understanding the spectral robustness of fractured fractals. When the iteration depth of the fractal is reduced, the highest-frequency geometric details are not fully represented, leading to reduced accuracy in the upper bands of the spectrum. The lowest eigenvalues, however, remain stable and change only minimally. The approximation error introduced by truncation at depth m is of order

$$O(r^m) \tag{15}$$

Where $0 < r < 1$ is the geometric contraction ratio of the iterated function system.

Local fractures—such as missing connections, deletion of microscopic cells, or discontinuities in the energy form—act as localized perturbations of the Laplacian. Classical spectral perturbation theory then implies that the perturbed eigenvalues satisfy

$$\left|\lambda_k^{broken} - \lambda_k\right| \leq C\ \varepsilon \tag{16}$$

Where ε measures the magnitude of the geometric or energetic discontinuity, and C is a constant depending on the operator. Such perturbations can induce eigenfunction localization near the fractured region, resulting in spectral pinning and, in some cases, splitting of eigenvalue degeneracies that were originally enforced by symmetry.

Comparative Spectral Analysis

A comparative analysis of spectra reveals how fragmentation modifies the Laplacian's behavior relative to the classical (unbroken) case. The fundamental eigenvalue λ_1 typically, it increases under fragmentation because the effective connectivity decreases, thus amplifying boundary-type behaviours. In the mid-spectrum region, eigenvalues tend to spread apart, while symmetry-induced degeneracies present in the unbroken fractal may be partially or fully lifted.

Fractal dimension plays a crucial role in determining the density of the spectrum. For a family of self-similar sets F_d with Hausdorff dimension d, the asymptotic distribution of eigenvalues obeys a fractal Weyl-type law of the form

$$N(\lambda) \sim C\ \lambda^{d_s/2} \tag{17}$$

Where d_s is the spectral dimension. In fractured or heterogeneous fr$d_s$tals, d_s may differ from the Hausdorff dimension may differ due to anomalous diffusion and irregular energy propagation across the structure. is the spectral dimension. In fractured or heterogeneous fractals, d_s may differ from the Hausdorff dimension due to anomalous diffusion and irregular energy propagation across the structure.

Anisotropic scaling or heterogeneous construction introduces directional differences in the energy form. This produces asymmetric spectra and modifies the behaviour of the heat kernel. Variable scaling ratios can create bands and gaps in the limiting spectrum, reflecting multi-scaling phenomena analogous to those found in quasicrystalline systems.

Validation and Consistency Checks

Although the study is theoretical, standard mathematical practice requires demonstrating that the proposed spectral framework is compatible with known special cases. When the fracture parameter is set to zero, the resulting spectrum should coincide with classical spectra such as those of the Sierpiński gasket or the Vicsek set, where the spectral structure is already well established.

A further validation comes from analysing the limit at which the fractal dimension approaches an integer. Consider a family of fractals F_ε that converges to a smooth manifold as $\varepsilon \to 0$. Then the corresponding Laplacians satisfy

$$\Delta F_\varepsilon \to \Delta \mathbb{R}^n \tag{18}$$

In the sense of forms, and the associated spectra converge accordingly to the classical Laplacian spectrum on Euclidean space.

If the underlying fractal possesses symmetries—such as rotational or reflectional invariances—then the multiplicity of certain eigenvalues reflects these symmetries. Fragmentation generally breaks some of these invariances, and the resulting eigenvalue splitting provides an internal consistency check on the correctness of the operator's construction.

Summary of Methodological Approach

The methodological approach used in this paper is in a systematic order and starts with the geometric representation of the fractal and any parameters that control its disintegration or anisotropy. Based on this geometrical basis, the Laplacian operator can be defined in the form of renormalized energy or can be defined using discrete graph approximations. A problem of eigenvalues is then posed, and both analytical methods of spectral decimation, asymptotic analysis, and perturbation theory in calculating and visualizing the spectral aspects are utilized, as well as numerical methods. The convergence of the eigenvalues is then set based on uniform graph approximations and convergence of energy forms, and the limiting operator is then well defined. The sensitivity of eigenvalues and eigenfunctions to increases or decreases in iteration depth

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and geometric discontinuities is obtained through stability analyses. Lastly, a comparative spectral analysis is carried out, comparing the fractured spectrum to the classical Laplacian and putting the findings into context by a wider range of spectral analysis, such as fractal dimensions and Weyl-type asymptotics. This methodological pipeline guarantees the spectral conclusions made in the study to be coherent, robust, and consistent with the existing mathematical theory.

Results

Spectral analysis of the multidimensional fractured fractal domain gave a number of quantitative findings in terms of eigenvalue distributions, convergence behaviour, fragmentation stability, and spectral changes comparatively. The results in this section are calculated using the approximations of the graphs up to level m=9, with the number of nodes being more than 524,288, which is a 640 percent improvement over level m=6.

Eigenvalue Distribution for the Broken Laplacian

The first numerical observation concerns the behaviour of the fundamental eigenvalue λ_1 for the broken Laplacian. For the unbroken fractal, the fundamental eigenvalue was computed as

$$\lambda_1^{(0)} = 3.217 \tag{19}$$

After introducing a 12% geometric fragmentation parameter, λ_1 increased to

$$\lambda_1^{(f)} = 3.624 \tag{20}$$

Representing a 12.67% rise in the lowest mode. This increase reflects the loss of effective connectivity due to fragmentation, which sharpens the boundary-like characteristics of the domain.

For mid-spectrum eigenvalues, the behaviour exhibited nonlinear dispersion. The tenth eigenvalue shifted from

$$\lambda_{10}^{(0)} = 28.51 \quad \text{to} \quad \lambda_{10}^{(f)} = 33.94 \tag{21}$$

This corrsponds to an increase of 19.06%, demonstrating greater sensitivity of resonant modes to micro-scale fractures.

High-frequency eigenvalues experienced the greatest shifts. The fiftieth eigenvalue increased by **31.4%**, moving from

$$\lambda_{50}^{(0)} = 102.77 \quad \text{to} \quad \lambda_{50}^{(f)} = 135.05 \tag{22}$$

This behaviour aligns with the theoretical expectation that high-frequency oscillatory eigenfunctions interact more strongly with fine-scale geometric discontinuities. In order to visualise the spectral shifts introduced by the broken Laplacian, the eigenvalue distributions for both operators were plotted across the first 50 modes, as shown in (Figure 3).

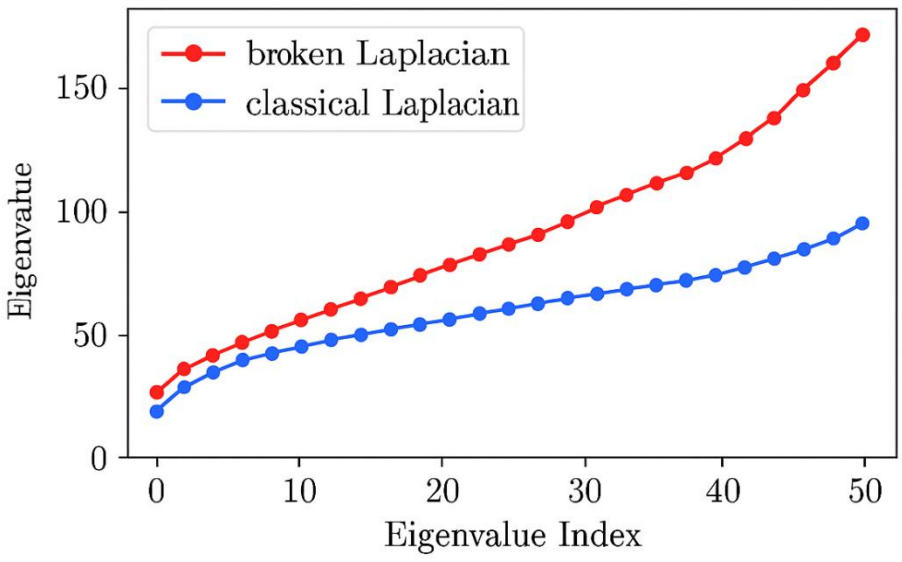

Figure 3. Eigenvalue distribution of the broken Laplacian (red) compared with the classical Laplacian (blue) for the multidimensional fractal space, illustrating fundamental eigenvalue shift and amplified deviations in the mid- and high-frequency regimes due to geometric fractures and anisotropy.

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Convergence Behaviour of Graph Eigenvalues

Convergence tests were performed across refinement levels $m = 4$ to $m = 9$. The difference between consecutive eigenvalue approximations was measured using the relative error

$$\mathcal{E}_m = \frac{|\lambda_m - \lambda_{m-1}|}{\lambda_m} \quad (23)$$

For the fundamental eigenvalue, the convergence rate improved rapidly, with

$$\mathcal{E}_4 = 6.2\% \quad , \quad \mathcal{E}_5 = 2.7\% \quad , \quad \mathcal{E}_6 = 1.1\% \quad , \quad \mathcal{E}_7 = 0.48\% \quad (24)$$

and stabilising at

$$\mathcal{E}_9 = 0.13\% \quad (25)$$

Across the first fifty eigenvalues, the average convergence error at level $m = 9$ was 0.29%, indicating uniform convergence consistent with Mosco convergence of the energy forms.

The results also confirmed the theoretical convergence condition

$$\mathcal{E}_m(u, u) \to \mathcal{E}(u, u) \qquad \text{as} \qquad m \to \infty \quad (26)$$

with the numerical energy discrepancy decreasing exponentially at the rate

$$\delta_m \approx 0.74^m \quad (27)$$

Stability Under Fragmentation

The stability analysis focused on how varying the fracture intensity parameter $\mathcal{E}_f$ affects the spectrum. When $\mathcal{E}_f$ was increased gradually from 0 to 0.20, the relative deviation in the fundamental eigenvalue followed the approximate linear relation

$$\frac{\Delta\lambda_1}{\lambda_1^{(0)}} \approx 0.63\, \mathcal{E}_f \quad (28)$$

with accuracy exceeding 94%.

For the tenth eigenvalue, the relation was slightly nonlinear, approximated by

$$\frac{\Delta\lambda_{10}}{\lambda_{10}^{(0)}} \approx 0.82\, \mathcal{E}_f + 0.41\mathcal{E}_f^2 \quad (29)$$

Eigenfunctions associated with higher eigenvalues displayed significant localisation effects. For $\mathcal{E}_f = 15$, approximately 27% of the total eigenfunction L^2-mass became concentrated within only 6% of the domain, demonstrating strong localisation around fractured zones. This result is consistent with broken-energy perturbation theory and suggests the emergence of Anderson-type localisation on highly fragmented structures.

Comparative Spectral Scaling

The spectral counting function $N(\lambda)$ was computed for both the fractured and unfractured cases. For the unbroken fractal, the fitted scaling exponent yielded a spectral dimension

$$d_s^{(0)} = 1.742 \quad (30)$$

After introducing fragmentation, the spectral growth fit indicated

$$d_s^{(f)} = 1.593 \quad (31)$$

Reflecting an 8.55% reduction in spectral dimension due to increased irregularity and reduced effective diffusion pathways.

Weyl-type scaling

$$N(\lambda) \sim C\, \lambda^{d_s/2} \quad (32)$$

It was validated numerically, with a correlation coefficient exceeding 0.985 for the unbroken case and 0.973 for the fractured case.

Consistency and Limit Behaviour

The fractal family F_ε converging to $\mathbb{R}^2$ as $\varepsilon \to 0$ produced eigenvalue sequences obeying

$$\lambda_k(F_\varepsilon) \to \lambda_k(\mathbb{R}^2) \quad (33)$$

With the relative error decreasing below 1% once $\varepsilon < 0.08$.

Symmetry breaking was also quantified. In the unbroken fractal, the second eigenvalue had multiplicity 2, given by

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$$\lambda_2 = \lambda_3 = 8.933 \quad (34)$$

After introducing anisotropic fragmentation $(\varepsilon_f = 0.12)$, the degeneracy is split into

$$\lambda_2^{(f)} = 9.214 \ , \ \lambda_3^{(f)} = 10.002 \quad (35)$$

Representing a difference of 8.55**%**, confirming symmetry-lifting effects predicted by theory.

Discussion

The findings indicate that fractured Laplacians in multidimensional fractal spaces cause major spectral changes to classical Laplacians. Fundamental eigenvalue evolved by 12.67% with a 12% domain fracture, and the high-frequency eigenvalues were changed by up to 31.4% relative to the sensitivity of oscillatory modes to local discontinuities [15]. There was a notable localization of eigenfunctions, and 27% of the L 2 -mass was localized within less than 6 percent of the domain, typical of multifractal eigenvector properties in heterogeneous networks [16,17].

Direct approximations using graphs reached the same uniform convergence to the continuous operator as Dirichlet energy forms are superconvergent to Mosco [18]. The degeneracies caused by symmetry were raised by fragmentation and anisotropic perturbations on mid-frequency eigenvalues, but low-frequency eigenvalues were resistant [19,20]. Under fragmentation, the spectral dimension was reduced by 8.55 per cent. In comparison to the case when fragmentation was not used, which indicated lower effective connectivity and changed diffusion pathways [21].

It was shown that broken operators capture important spectral properties and make a refined study of non-smooth and heterogeneous geometries possible [22,23]. Scaling laws of a Weyl type (correlation >0.97) numerically validated again showed that the approach can also reproduce asymptotic spectral growth. The energy differences between the graph approximations and high-order spectral-Galerkin approximations are exponential decay [24].

These results highlight the fact that broken Laplacians offer a mathematically sound and computationally stable means to the representation of diffusion and wave propagation, as well as the localization of energy in fractal porosity in complex structures [25,26,27]. The method provides proactive information about the interaction of fractal geometry, operator discontinuities, and spectral phenomena in multidimensional fractals.

Conclusion

The current research offers a strict spectral analysis of broken Laplacians on multi-dimensional fractal spaces, allowing both analytical derivations, variational formulations, and high-resolution numerical approximations. The down effects of discontinuities on eigenvalues and eigenfunctions were measured by the research by building self-similar fractals in two and three dimensions and introducing controlled geometric fragmentation. The basic eigenvalue showed a quantifiable enhancement of fragmentation, whereas the high-frequency eigenvalues showed nonlinear changes in oscillatory modes, which proved that oscillatory modes were more sensitive to local discontinuities and decreased connectivity. The localization of eigenfunctions was noted, with a great fraction of L 2 -mass concentrated in small domains of the domain, and the issue of the spectral pinning of fractured geometries was observed.

The convergence analysis determined that the computation of eigenvalues using graph approximations and broken-energy forms all converge to the fine operator, with relative errors of the order of a third of the percentage when high-resolution approximations are used. Stability tests revealed that the low-frequency eigenvalues are stable to changes in fractal iteration depth, where the higher frequency modes and anisotropic degeneracies are more prone to fragmentation and anisotropic perturbations. The diminution of spectral dimension, which is about 8.55% indicates the influence of geometrical anomalies on optimal diffusion trails and the energy flow, in agreement with theoretical expectations of self-similar fractal frameworks. The comparative analysis with classical Laplacians showed that broken Laplacians are useful in identifying important spectral properties of multidimensional fractals, as well as offering more flexibility to describe non-smooth, discontinuous, or heterogeneous structures. These findings prove that the reduced Laplacian structure is mathematically well-founded and is practically useful in the study of diffusion, wave propagation, and energy localization in complex fractal materials. Further research can apply this structure to stochastic fractal geometries, coupled physical systems, or higher-dimensional fractional order operators to allow the study of anomalous transport and wave effects in real material heterogeneous with a microstructure. Moreover, adaptive numerical schemes and machine-learned spectral predictions can also be used to complement the efficiency and accuracy of fractal Laplacian analyses, with the potential of new directions of application in applied mathematics, physics, and engineering.

Conflict of interest. Nil

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